Higher structures in quantum gauge theories

Joint work with A. Schenkel (Nottingham) & L. Woike (Hamburg)

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

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Universität Hamburg
Fachbereich Mathematik

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Plan

- Higher information in gauge theories
- Using it! $\rightsquigarrow$ BRST-BV
- Function algebras and homotopy algebraic QFT
- Homotopy AQFTs from AQFT-invariants

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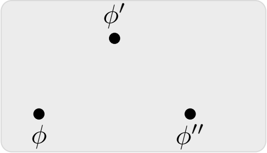
Bottomline: Elementary gauge theoretic constructions suggest a homotopical enhancement of the AQFT axioms.

Disclaimer!
The following **informal presentation** focuses on examples/ideas.
For actual statements see arXiv:1709.08657 & 1805.08795
and for a review refer to arXiv:1903.02878.

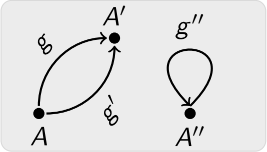
Higher structures in gauge theories

Ordinary fields



Information captured by ϕ's

Gauge fields



Two layers of information:

(1) Gauge classes $[A]$

(2) For each A, the g's preserving it

Higher information! (groupoid)

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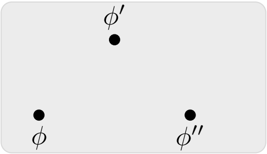

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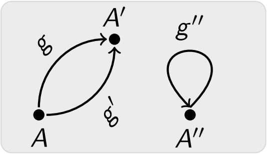

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- Compare Yang-Mills with structure groups $\mathbb{R}$ and $U(1)$:
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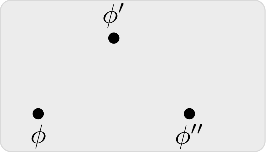

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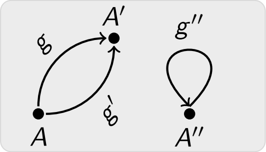

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- Compare Yang-Mills with structure groups $\mathbb{R}$ and $U(1)$:
  On Minkowski (1) is the same for both, but (2) is $\mathbb{R}$ vs $U(1)$!
- (2) plays a crucial role in the BRST-BV formalism...

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- Mechanical systems and ordinary fields ✓
- Analogue in gauge theory requires appropriate framework…

Going on-shell in gauge theory: Set-up

Consider the "action" $S = \int_M dA \wedge *dA$ for the vector potential A on a spacetime M. Linearity $\implies$ configurations as a chain complex:

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We are trying to imitate the standard process for going on-shell, so the first attempt is to take the intersection of dS with 0:

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How to achieve this?
General recipe from homological algebra $\rightsquigarrow$ *derived functors*

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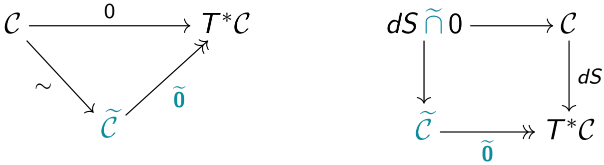

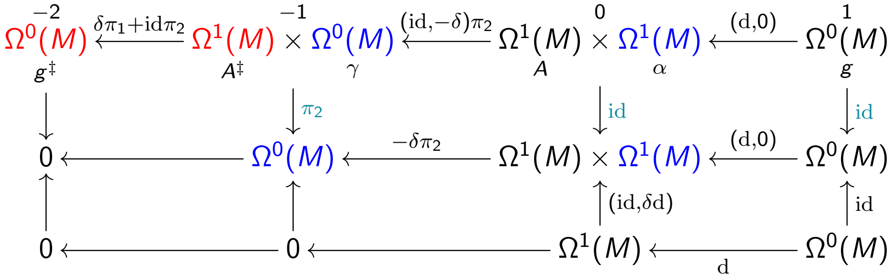

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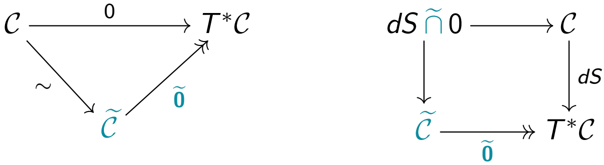

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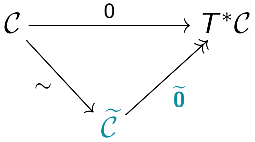

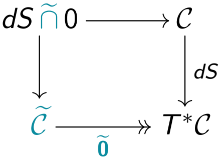

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Hence: linear gauge theory $(A, g, \dots)$ $\rightsquigarrow$ chain complex
$\rightsquigarrow$ quasi-isomophisms $\rightsquigarrow$ derived intersection $\rightsquigarrow$ BRST-BV

Classical observables

Classical observables := $\mathbb{C}$-valued functions on configuration space:

$$\mathcal{C} \quad \longmapsto \quad \mathit{Map}(\mathcal{C}, \mathbb{C})$$

Commutative algebra! (quantization $\rightsquigarrow$ non-commutative)

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Tipical situation in field theory:

$\mathcal{C} : M \mapsto \mathcal{C}_M$ *contravariant* assignment
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Let us try to imitate this construction with a gauge theory...

Configuration space of a gauge theory

The data of a gauge theory are efficiently captured by simplicial sets:

$$\underset{\text{gauge fields}}{\mathcal{C}_0} \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \underset{\text{gauge trafos}}{\mathcal{C}_1} \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \underset{\text{higher trafos}}{\mathcal{C}_2} \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \cdots$$

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Prototypical example: G-bundles with connection on $M \cong \mathbb{R}^m$

$$\Omega^1(\underset{A}{M}, \mathfrak{g}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \Omega^1(\underset{A}{M}, \mathfrak{g}) \times C^\infty(\underset{g_0}{M, G}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \Omega^1(\underset{A}{M}, \mathfrak{g}) \times C^\infty(\underset{(g_0,g_1)}{M, G})^{\times 2} \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \cdots$$

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- Contravariant in M $\Longrightarrow$ covariant passing to function algebras

We need function algebras on simplicial sets...

Classical observables for gauge theories

Homotopy theory provides function algebras on simplicial sets:

1. Take $\mathbb{C}$-valued functions on each level of the simplicial set $\mathcal{C}$:

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2. Form associated complex $N^*(\mathcal{C})$ of normalized cochains on $\mathcal{C}$:

$$Map(\mathcal{C}_0,\mathbb{C}) \longrightarrow Map_N(\mathcal{C}_1,\mathbb{C}) \longrightarrow Map_N(\mathcal{C}_2,\mathbb{C}) \longrightarrow \cdots$$

(normalizazion $\leftrightarrow$ left-pointing arrows & differential $\leftrightarrow$ right-pointing arrows)

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$\rightsquigarrow$ First instance of **homotopy A(Q)FT**

Algebraic quantum field theory (AQFT)

Broad spectrum of concepts with similar behavior:

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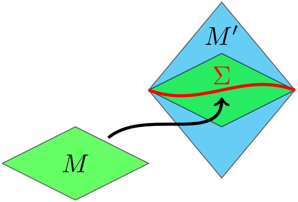
subject to additional axioms, such as causality, time-slice, ...
reflecting physically motivated features.

Time-slice axiom

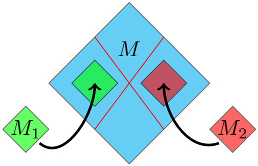


$\{$Cauchy morphisms$\} \subseteq$ Loc

$\mathcal{A}(f) : \mathcal{A}(M) \longrightarrow \mathcal{A}(M')$
is an isomorphism

Dynamical law:
Theory on a spacetime determined by algebra of observables on a neighborhood of a Cauchy surface.

Causality



For causally disjoint spacetimes

$$[\mathcal{A}(f_1)-, \mathcal{A}(f_2)-] = 0$$

Nothing travels faster than light:
Spacelike separated observables act independently of each other since the algebras associated to causally disjoint regions commute in the ambient.

Invariants of AQFTs: Idea

Starting point:

AQFT $\mathcal{A} : \mathsf{Str} \to \mathsf{Alg}$ *on spacetimes with additional background data*

$$\begin{array}{ccc} \mathsf{Str} & \xrightarrow{\mathcal{A}} & \mathsf{Alg} \\ {\scriptstyle \text{forget}} \big\downarrow & & \\ \mathsf{Loc} & & \end{array}$$

Examples:

- Dirac fields – spin structures
- Fields coupled to background gauge fields – bundles + connection
- Global gauge symmetries
- External sources

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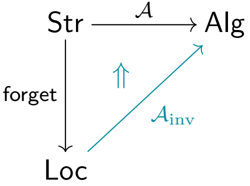

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Goal: Invariant QFT $\mathcal{A}_{\mathrm{inv}} : \mathsf{Loc} \to \mathsf{Alg}$ *independent of structures*

Invariants of AQFTs: Construction and properties

Computing invariants on a spacetime M:

$S \in \pi^{-1}(M)$ structures on M

$h : S \to S'$ isomorphisms of such structures (symmetries)

$$\mathcal{A}_{\mathrm{inv}}(M) := \Big\{ a \in \prod_{S \in \pi^{-1}(M)} \mathcal{A}(S) : \quad \mathcal{A}(h) a(S) = a(S') \;\; \forall h : S \to S' \Big\}$$

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Interpretation: $a \in \mathcal{A}_{\mathrm{inv}}(M)$ is a function assigning to each structure S over M an observable $a(S) \in \mathcal{A}(S)$. The condition ensures that $S \mapsto a(S)$ is invariant under symmetries of S's.

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- Time-slice: ✓ (mild assumptions on $\pi : \mathsf{Str} \to \mathsf{Loc}$)
- Isotony: ✗ (requires very restrictive hypotheses)

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$$\begin{aligned} d_1 f(x_0, x_1) &= f(x_0) - f(x_0 + x_1) + f(x_1) \\ d_2 \omega(x_0, x_1, x_2) &= \omega(x_0, x_1) - \omega(x_0 + x_1, x_2) + \omega(x_0, x_1 + x_2) - \omega(x_1, x_2) \\ \ldots &= \ldots \end{aligned}$$

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The non-trivial 2-cocycle $\omega \in \ker d_2$, $\omega(x_0,x_1) := x_0^1\, x_1^2$, defines the *Heisenberg group* $\rightsquigarrow$ **Interesting info in higher cohomologies!**

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If we "combine" the construction of invariants with the previous example, for each spacetime M, we get a simplicial set

$$\underbrace{\{S_0 \in \pi^{-1}(M)\}}_{\text{structures over } M} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \underbrace{\{S_0 \xleftarrow{h_0} S_1 \text{ over } \mathrm{id}_M\}}_{\text{symmetries}} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \underbrace{\{S_0 \xleftarrow{h_0} S_1 \xleftarrow{h_1} S_2\}}_{\text{higher...}} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots$$

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$$\mathcal{A}_{\mathrm{hinv}}(M) := \Big(\prod_{S_0} \mathcal{A}(S_0) \xrightarrow{d_0} \prod_{S_0 \overset{h_0}{\leftarrow} S_1} \mathcal{A}(S_0) \xrightarrow{d_1} \prod_{S_0 \overset{h_0}{\leftarrow} S_1 \overset{h_1}{\leftarrow} S_2} \mathcal{A}(S_0) \xrightarrow{d_2} \cdots \Big)$$

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An n-cochain a looks like an $\mathcal{A}$-valued map

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The differential of the n-cochain a is computed by

$$d_n a(h_0, \dots, h_n) = (-1)^0 a(h_0, \dots, h_{n-1}) + \sum_{i=1}^{n} (-1)^i a(\dots, h_{n-i} h_{n-i+1}, \dots) + (-1)^{n+1} \mathcal{A}(h_0) a(h_1, \dots, h_n)$$

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$\rightsquigarrow$ Another instance of **homotopy AQFT**

Causality vs causality up to coherent homotopies

Given an AQFT $\mathcal{A}$, two causally disjoint embeddings

$$M_1 \xrightarrow{f_1} M \xleftarrow{f_2} M_2$$

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We need more sophisticated techniques...

Formalizing homotopy AQFTs

New structure emerging from examples

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- Gauge theories carry higher structures (e.g. encoded in chain complexes)

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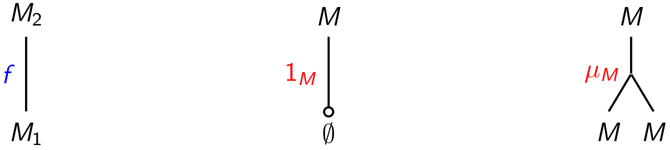

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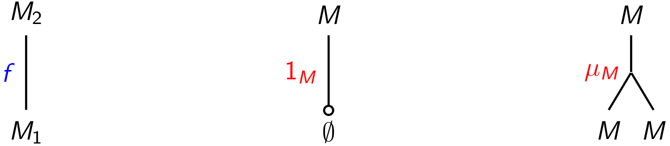
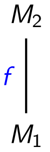
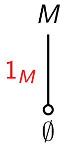


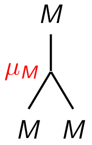
Structural identities: **functor** + associative + compatible + *causal*

The essence of AQFT

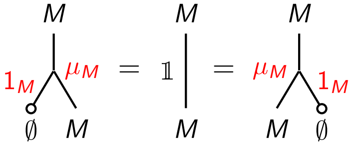
Elementary operations: push-forward + multiply

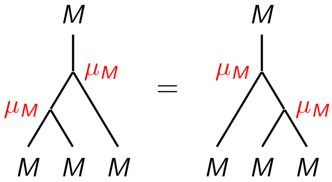






Structural identities: functor + **associative** + compatible + *causal*





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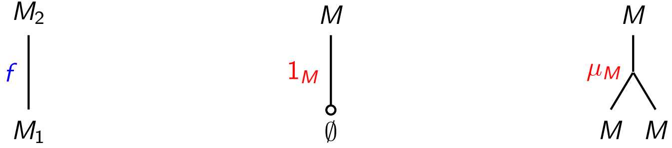

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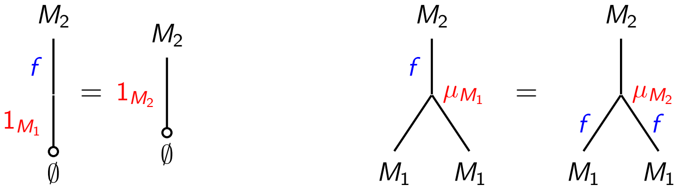

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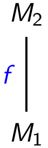

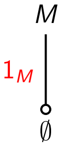

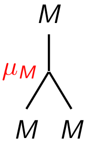

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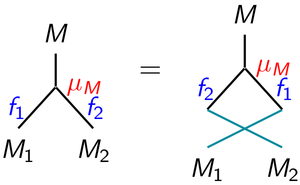
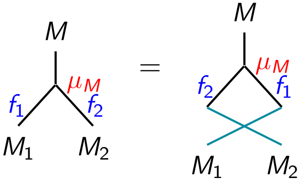


for f_1, f_2 causally disjoint

A glimpse of homotopy causality

Instead of the **identity**



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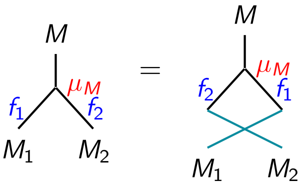
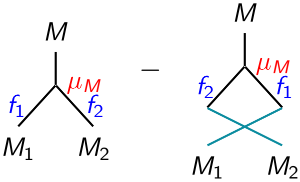


for f_1, f_2 causally disjoint

introduce a 1-chain η_{f_1,f_2} such that



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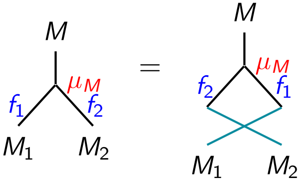

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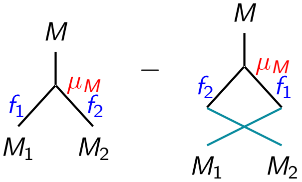

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Now go higher (both in degree and arity) $\rightsquigarrow$ Homotopy theory of operads!